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Note on Proizvolov's example

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Let f be a map (= continuous map) of a topological space X onto a topological space Y. We say that f is a <u>compact-covering map</u> if every compact subset of Y is the image of some compact subset of X under f. In the following cases, every open map is compact-covering :

(1) (E. Michael [4]) X is a metric space and Y is a T_2 space and, for some metric on X, $f^{-1}(y)$ is complete for each $y \in Y$.

(2) (A. Arhangel'skii [2]) X is a Čech-complete space and Y is a T_2 space.

(3) (K. Alster [1]) X is a metric space and Y is a countable T_2 space.

(4) (K. Nagami [5]) X is a p-space and Y is a T_3 space, and $f^{-1}(y)$ is compact for each $y \in Y$.

On the other hand, V.V.Proizvolov [6] constructed an example such that there exists an open, at most two-to-one map from a Lindelöf, first countable T_3 space onto a compact metric space, which is not compact-covering.

In this note, we would like to give an adding explanation for his example, using the following lemma :

<u>Lemma.</u> <u>Let</u> $(X, \mathfrak{T}_1)$ <u>and</u> $(X, \mathfrak{T}_2)$ <u>be compact</u> T_2 <u>spaces with</u> $\mathfrak{T}_1 \subset \mathfrak{T}_2$. <u>Then</u> $\mathfrak{T}_1 = \mathfrak{T}_2$ <u>holds.</u>

This is an immediate consequence from the fact that the identity map from $(X, \mathfrak{T}_2)$ to $(X, \mathfrak{T}_1)$ is homeomorphic.

<u>Proizvolov's example.</u> Let P be the set $[0,1]\times[0,1]$ and $P_0 = [0,1]\times\{\frac{1}{2}\}$, and define the topology of P as below : If $p \in P - P_0$, p has a neighborhood base in the usual sense of Euclidean plane. For any $p = (p_1, \frac{1}{2}) \in P_0$ and for any natural numbers ℓ, m and n, let $U_{\ell mn}(p)$ be the subset of P which consists of p and of all points satisfying one of the following three conditions : (1) $p'_2 \leq \frac{1}{2}$, and $(p_1+\frac{1}{n}-p'_1)^2+(\frac{1}{2}-p'_2)^2 < \frac{1}{n^2}$ or $(p_1-\frac{1}{n}-p'_1)^2+(\frac{1}{2}-p'_2)^2 < \frac{1}{n^2}$; (2) $p_1-\frac{2}{n} < p'_1 < p_1+\frac{2}{n}$, $\frac{1}{2} \leq p'_2$ and $p'_2-\frac{1}{2} < \frac{1}{m}|p'_1-p_1|$; (3) $\frac{1}{2} \leq p'_2 < \frac{1}{2}+\frac{1}{\ell}$ and $p'_2-\frac{1}{2} > m|p'_1-p_1|$, , and let $\{U_{\ell mn}(p)\}_{\ell,m,n=1}^{\infty}$ be the neighborhood base at p. Then it is easily seen that P is a Lindelöf, first countable T_3 space. Let $Y = [0,1]\times[0,\frac{1}{2}]$ be the subspace of

the Euclidean plane and f the map from P onto Y such that it identifies the points which are symmetric with respect to P_0. Then f is clearly an open, at most two-to-one map. It remains to show that f is not compact-covering. On the contrary, suppse f is compact-covering. Then there exists a compact subset K of P, whose image by f covers Y. Let $\mathfrak{T}_1$ be the topology of K as the subspace of the Euclidean plane, and let $\mathfrak{T}_2$ be the topology of K as the subspace of P. Then $\mathfrak{T}_1 < \mathfrak{T}_2$ holds. Hence, by Lemma $\mathfrak{T}_1 = \mathfrak{T}_2$ holds. On the other hand, since K covers P_0, by the definition of $\mathfrak{T}_2$ it contains no countable base ; however, $\mathfrak{T}_1$ contains a countable base. This contradiction shows that f is not compact-covering.

Supplement. A space X is called a space of countable (resp. point-countable) type if every compact subset (resp. point) of X is contained in some compact subset of X with a countable neighborhood base (cf. [3]).

As for the research of K. Nagami [5], there was a question whether every open compact map defined on a T_3 space of countable type is compact-covering, and it was informed that V. V. Proizvolov [6] solved it in the negative. However, in his example mentioned above, P could not be of countable type.

Because, P_o is a compact G_δ-subset of P which has no countable neighborhood base in P. Hence, it seems that such question remains still open.

Refernces

1. K. Alster, Remaks on compact-covering mappings, Bull. de l'Acad. Polonaise des Sci., XIX (1971), 141--148.

2. A. V. Arhangel'skii, Open and near open mappings. Connections between spaces, Trudy Moskov.Math. Obshch., 15 (1966), 181--223.

3. A. V. Arhangel'skii, On a class of spaces containing all metric and all locally bicompact spaces, Dokl. Acad. Nauk SSSR, 151 (1963), 751--754 ; Soviet Math. Dokl., 4 (1963), 1051--1055.

4. E. A. Michael, A theorem on semi-continuous set-valued functions, Duke Math. J., 26 (1959), 647--652.

5. K. Nagami, Minimal class generated by open compact and perfect mappings, Fund. Math., LXXVIII (1973), 227--264.

6. V. V. Proizvolov, Solution answering a problem of Michael about k-covering mappings, Proc. 2nd Tiraspol Sympos on General Topology and its Applications, Kishinev 1969, 62 --64.